ECE 346: Digital Signal Processing — Spring 2021
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Recitation #1: Review of Mathematical Concepts

Objective & Outline

The objective of this week's recitation section is to review some basic mathematical concepts that we will use over and over again throughout this course. Mastering (or at least getting good at) these topics will help you tremendously in the long run. The following is the outline of this solution guide:

1. Problems $1-4$: recitation problems
2. Problem 5: self-assessment problem

Problem 1 (Operations with Complex Exponentials)**.** Perform the following operations (if necessary) and simplify each of the signals into sinusoidal form using Euler's formulas (e.g. $\mathrm{e}^{j\theta} = \cos(\theta) + j\sin(\theta)$).

(a) $x_1(t) = 3\mathrm{e}^{j\frac{\pi}{3}t} \cdot 4\mathrm{e}^{j\frac{9\pi}{4}t}$

(b) $x_2(t) = \frac{2\mathrm{e}^{j\frac{\pi}{2}t}}{4\mathrm{e}^{j\pi t}}$

(c) $x_3(t) = \frac{1-\mathrm{e}^{-j\pi t}}{1+\mathrm{e}^{-j\pi t}}$

Solution:

(a) We want to first multiply the two complex exponentials and then express them in terms of sinusoids using Euler's formula:

$$\begin{aligned} x_1(t) &= 3\mathrm{e}^{j\frac{\pi}{3}t} \cdot 4\mathrm{e}^{j\frac{9\pi}{4}t} && (1)\\ &= 12\mathrm{e}^{j(\frac{\pi}{3}+\frac{9\pi}{4})t} && (2)\\ &= 12\mathrm{e}^{j\frac{31\pi}{12}t} && (3)\\ &= 12\mathrm{e}^{j\frac{7\pi}{12}t} && (4)\\ &= 12\cos\left(\frac{7\pi}{12}t\right) + 12j\sin\left(\frac{7\pi}{12}t\right). && (5) \end{aligned}$$

(b) We use the same process for this problem but with division:

$$x_2(t) = \frac{2e^{j\frac{\pi}{2}t}}{4e^{j\pi t}} \tag{6}$$
$$= \frac{1}{2}e^{-j\frac{\pi}{2}t} \tag{7}$$
$$= \frac{1}{2}\cos\left(-\frac{\pi}{2}t\right) + \frac{1}{2}j\sin\left(-\frac{\pi}{2}t\right) \tag{8}$$
$$= \frac{1}{2}\cos\left(\frac{\pi}{2}t\right) - \frac{1}{2}j\sin\left(\frac{\pi}{2}t\right) \tag{9}$$

Note that to get to the final answer, we used facts regarding even and odd functions.

(c) This problem is a little trickier in the sense that you shouldn't try to simplify this by dividing. Instead, we can factor out a common value and then use Euler's formulas.

$$x_3(t) = \frac{1 - e^{-j\pi t}}{1 + e^{-j\pi t}} \tag{10}$$
$$= \frac{e^{j0} - e^{-j\pi t}}{e^{j0} + e^{-j\pi t}} \tag{11}$$
$$= \frac{e^{-j\frac{\pi}{2}t}}{e^{-j\frac{\pi}{2}t}}\left[\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}\right] \tag{12}$$
$$= \frac{2j\sin(\frac{\pi}{2}t)}{2\cos(\frac{\pi}{2}t)} \tag{13}$$
$$= j\tan\left(\frac{\pi}{2}t\right). \tag{14}$$

We can actually simplify this function further by defining what values $x_3(t)$ takes at different values of t, but this answer is sufficient.

Problem 2 (Complex Signals)**.** For each of the complex signals, calculate the magnitude and phase of the functions analytically.

(a) $x_1(t) = e^{j(\frac{t}{2}+3)}$

(b) $x_2(t) = e^{-j\frac{t}{2}} + 2e^{-j\frac{t}{3}}$

Solution:

(a) We can separate the signal into its real and imaginary parts and do the following calculations:

$$\mathfrak{Re}\{x_1(t)\} = \cos\left(\frac{t}{2}+3\right) \tag{15}$$

$$\mathfrak{Im}\{x_1(t)\} = \sin\left(\frac{t}{2}+3\right) \tag{16}$$

$$|x_1(t)| = 1 \tag{17}$$

$$\measuredangle x_1(t) = \tan^{-1}\left(\tan\left(\frac{t}{2}+3\right)\right) \tag{18}$$

$$= \frac{t}{2}+3 \tag{19}$$

(b) Same procedure except our signal becomes a little more complicated:

$$\mathfrak{Re}\{x_2(t)\} = \cos(t/2) + 2\cos(t/3) \tag{20}$$

$$\mathfrak{Im}\{x_2(t)\} = -\sin(t/2) - 2\sin(t/3) \tag{21}$$

$$|x_2(t)| = \left((\cos(t/2) + 2\cos(t/3))^2 + (-\sin(t/2) - 2\sin(t/3))^2\right)^{1/2} \tag{22}$$

$$\measuredangle x_2(t) = \tan^{-1}\left(\frac{-\sin(t/2) - 2\sin(t/3)}{\cos(t/2) + 2\cos(t/3)}\right) \tag{23}$$

Problem 3 (Sifting Property of Impulses)**.** Recall that the "sifting property of impulses" says that integrating a signal multiplied by an impuse returns the value of the function at the location of the impulse:

$$\int_a^b x(t)\delta(t-c)dt = x(c). \tag{24}$$

For each of the signals, compute the integral.

(a) $\int_{-\infty}^{\infty} t^2\delta(2t-4)$

(b) $\int_{-\infty}^{\infty} \cos(t)\delta(4t-\pi)$

(c) $\int_1^5 t^3\delta(t+2)$

Solution:

By using equation (24) we can easily evaluate these integrals.

(a)

$$\delta(2t-4) = \frac{1}{2}\delta(t-2) \tag{25}$$

$$\int_{-\infty}^{\infty} t^2\delta(2t-4) = \frac{1}{2}(2)^2 \tag{26}$$

$$= 2. \tag{27}$$

(b)

$$\delta(4t-\pi) = \frac{1}{4}\delta(t-\frac{\pi}{4}) \tag{28}$$

$$\int_{-\infty}^{\infty} \cos(t)\delta(4t-\pi) = \frac{1}{4}\cos\left(\frac{\pi}{4}\right) \tag{29}$$

$$= \frac{\sqrt{2}}{8}. \tag{30}$$

(c) This one is a little trickier. Note that $\delta(t+2)$ has a time at $t=-2$. Since $t=-2$ is out of our integral bounds $(1 < t < 5)$,

$$\int_{1}^{5} t^3\delta(t+2) = 0 \tag{31}$$

Problem 4 (Energy of Signals)**.** Recall that the energy of a DT signal, say $x[n]$, is

$$\sum_{-\infty}^{\infty} |x[n]|^2. \tag{32}$$

Compute the energy of each signal.

(a) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

(b) $x_2[n] = \left(\frac{1}{3}\right)^n u[n]$

(c) $x_3[n] = \left(\frac{2}{3}\right)^{n+2} u[n]$

Solution:

The main purpose of this problem is for you to use the geometric series formula. This formula will be used time and time again so you should remember it!

(a)

$$\mathcal{E}_1 = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u[n]\right|^2 \tag{33}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \tag{34}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \tag{35}$$

$$= \frac{1}{1-1/4} \tag{36}$$

$$= \frac{4}{3}. \tag{37}$$

(b)

$$\mathcal{E}_2 = \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{3}\right)^n u[n] \right|^2 \tag{38}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \tag{39}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^{n} \tag{40}$$

$$= \frac{1}{1-1/9} \tag{41}$$

$$= \frac{9}{8}. \tag{42}$$

(c)

$$\mathcal{E}_3 = \sum_{n=-\infty}^{\infty} \left| \left(\frac{2}{3}\right)^{n+2} u[n] \right|^2 \tag{43}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{4} \left(\frac{2}{3}\right)^{2n} \tag{44}$$

$$= \left(\frac{2}{3}\right)^{4} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{2n} \tag{45}$$

$$= \left(\frac{2}{3}\right)^{4} \frac{1}{1-4/9} \tag{46}$$

$$= \left(\frac{2}{3}\right)^{4} \frac{9}{5}. \tag{47}$$

Problem 5 (Self-assessment)**.** Try to solve each problem by yourself first, and then discuss with your group.

1. **Complex Exponentials.** Simplify the following signal into a sinusoid:

$$x[n] = (3\mathrm{e}^{j\theta})^n + (3\mathrm{e}^{-j\theta})^n \tag{48}$$

2. **Complex Signals.** Compute the magnitude and phase of the function

$$x[n] = 2\mathrm{e}^{j\frac{\pi}{5}n} \tag{49}$$

3. **L'Hopital's Rule.** Evaluate the following limits:

   (a) $\lim_{x\to 2} \dfrac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

   (b) $\lim_{x\to 2} \dfrac{\sin(\pi x)}{x^2 - 4}$

Solution:

1.

$$\begin{aligned} x[n] &= (3e^{j\theta})^n + (3e^{-j\theta})^n && (50) \\ &= 3^n e^{j\theta n} + 3^n e^{-j\theta n} && (51) \\ &= 3^n [e^{j\theta n} + e^{-j\theta n}] && (52) \\ &= 2 \cdot 3^n \cos(\theta n) && (53) \end{aligned}$$

2.

$$\begin{aligned} \mathfrak{Re}\{x[n]\} &= 2\cos((\pi/5)n) && (54) \\ \mathfrak{Im}\{x[n]\} &= 2\sin((\pi/5)n) && (55) \\ |x[n]| &= 2 && (56) \\ \measuredangle x[n] &= \tan^{-1}\left(\tan\left(\frac{\pi}{5}n\right)\right) && (57) \\ &= \frac{\pi}{5}n && (58) \end{aligned}$$

3. (a)

$$\begin{aligned} \lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} &= \lim_{x\to 2} \frac{3x^2 - 14x + 10}{2x + 1} && (59) \\ &= -\frac{6}{5} && (60) \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x\to 2} \frac{\sin(\pi x)}{x^2 - 4} &= \lim_{x\to 2} \frac{\pi \cos(\pi x)}{2x} && (61) \\ &= \frac{\pi}{4} && (62) \end{aligned}$$